On the Generalisation of the Zipf-Mandelbrot Distribution and its Application to the Study of Queues with Heavy Tails

Demetres Kouvatsos

NetPEN

Networks and Performance Engineering Research Group, University of Bradford, UK
Outline

- Motivation: Info. Theory, Statistical Mechanics & Quantification Theory ➔ Queues with bursty & heavy tails

- Maximum Entropy (ME) Formalism
  - ME and GB Solutions for the State Probability Distributions of queues with bursty (GE-type) tails

- An Extended ME (EME) Formalism
  - EME Solutions for the State Probability Distributions of queues with heavy tails

- Numerical Experiments

- Conclusions and further remarks on the ME extension to the analysis of open QNMs
Motivation: Information Theory, Statistical Mechanics & Quantification Theory → Queues with Bursty & Heavy Tails

To consider alternative analytic methodologies for queues with bursty and heavy tails, based on a balanced trade-off between simplified assumptions to reduce complexity and actual real life system behaviour, leading to credible and cost-effective approximations for performance prediction and optimisation of telecommunication systems.
**Extended ME Formalism, Statistical Mechanics & Long-Range Interactions**

**In Statistical Mechanics:**

Energy are assumed to be

- "Extensive" variables
  - such as total energy $\rightarrow \sim$ system size
    (c.f., due to short-range interactions e.g., chemical bonds)

Similarly, entropy is also assumed to be extensive.

- "Non-extensive" variables
  - $\rightarrow$ energy no longer $\sim$ system size
    (c.f., due to long-range interactions such as gravity)

This makes life difficult in Statistical Mechanics!
Extended ME Formalism, Statistical Mechanics & Long-Range Interactions

- **Maximum Entropy (ME) Principle**

  \{\text{max Gibbs ‘Extensive’} \text{ Entropy Function,}

  subject to a \text{mean value constraint} of a quantity
  (e.g., system energy, \# of molecules, volume)\}

Applying Method of Lagrange Undetermined Multipliers

⇒ **Geometric** Steady State Prob. Distribution

(Lagrange multipliers are

“intensive” variables ⇔ “extensive” ones with constrained means

(e.g., energy ⇔ temperature, volume ⇔ pressure,

\# of molecules ⇔ chemical potential etc)
Extended ME Formalism, Statistical Mechanics & Long-Range Interactions

- **Generalised Maximum Entropy Principle**
  \{max the Havrda-Charvat ‘non-extensive’ entropy function\} (a quantitative measure of classification, subject to a mean value constraint)

- **Zipf-Mandelbrot** Steady State Prob. Distribution with power-law (heavy) tails and non-extensivity real-valued parameter \(q\)

The Zipf-Mandelbrot Distribution

The **Zipf-Mandelbrot** distribution is a discrete probability distribution. It is a power-law distribution on ranked data.

The probability mass function (pmf) is of the form

$$p(n, u, s) = \frac{(n + u)^{-s}}{\sum_{n=1}^{N} (n + u)^{-s}}$$

- **$N$** - the number of elements
- **$n, u$** - real numbers
- **$s$** - the value of the exponent characterizing the distribution
The Zipf-Mandelbrot Distribution

- In the limit as \( N \to \infty \), the sum \( \sum_{n=1}^{N} (n + u)^{-s} \)

becomes the Hurwitz-Zeta function \( \zeta(u, s) \)

- For finite \( N \) and \( u=0 \), the Zipf-Mandelbrot law becomes Zipf’s law (both commonly used in linguistics, Information Sciences, insurance, the modelling of events and ensemble theory in statistical mechanics)

- For infinite \( N \) and \( u=0 \), the sum is recognized as the Zeta distribution
The G/G/1 Queue & G/G/1/N Censored Queue with Bursty and/or LRD Traffic Flows

- A stable G/G/1 Queue
- A censored G/G/1/N Queue

\{\lambda, C^2_a, \mu, C^2_s, \alpha, H, N\}: the mean arrival rate and the interarrival sq. coef. of variation

H: Hurts parameter of the arrival process, N: Finite buffer capacity

\{\mu, C^2_s\}: mean service rate and sq. coef. of variation.
Maximum Entropy (ME) Formalism (Jaynes 1956a,b)

- System Specification
- Optimisation Problem Formulation
- Analytic Methodology
- ME Solution
- Basic Relations
- Overview of ME and Queueing Network Models (QNMs)
System Specification

- Q, General System;
- \( S = \{S_0, S_1, \ldots, S_n, \ldots\} \)
  Finite or countable infinite set of states;
- \( P(S_n) \), state prob. distr. that Q is at state \( S_n \);
- \( \{<f_k>\}, k=1, 2, \ldots, m < |Q|, \)
  Set of prescribed mean values defined on the set of suitable functions:
  \( \{f_1(S_n), f_2(S_n), \ldots, f_m(S_n)\} \)
Optimisation Problem Formulation

$$\max_P \left\{ H(P) = \sum_{S_n \in S} P(S_n) \log P(S_n) \right\}$$

subject to

$$\sum_{S_n \in S} P(S_n) = 1,$$

$$\sum_{S_n \in S} f_k(S_n)P(S_n) = \langle f_k \rangle, \quad k = 1, 2, \ldots, m$$

where $m$ is less than the number of possible states.

Apply the Method of Lagrange’s Undetermined Multipliers
\[ P(S_n) = \frac{1}{Z} \prod_{k=1}^{m} x_k f_k(S_n), \]

\[ Z = e^{\beta_0} = \sum_{S_n \in S} \prod_{k=1}^{m} x_k f_k(S_n), \]

Normalising Constant

\[ x_k = e^{-\beta_k}, \quad k = 1, 2, \ldots, m \]

\{\beta_k\} are the Lagrangian coefficients corresponding to constraints \{<f_k>\}, k=1, 2, \ldots, m
Basic Relations

\[ \frac{\partial \beta_0}{\partial \beta_k} = \langle f_k \rangle, \quad k = 1, 2, \ldots, m \]

\[ \max_P \{H(P)\} = \beta_0 + \sum_{k=1}^{m} \beta_k \langle f_k \rangle \]
OPEN QNM WITH JOINT STATE PROBABILITY

\{P(n), n = (n_1, n_2, \ldots, n_M), n_i \geq 0, n_i, 1,2, \ldots, M\}

CLASSICAL QUEUEING THEORY

MAX ENTROPY / EXTENDED ENTROPY

\[\text{Max}_p \ H(p)\]

ME FORMALISM & OPEN QNMs

PRODUCT FORM APPROXIMATION OF AN OPEN QNM

\[P(n) = \prod_{i=1}^{M} P_i(n_i)\]

INTERPRETATION

TRAFFIC FLOW ANALYSIS & ME QUEUE-BY-QUEUE DECOMPOSITION OF OPEN QNMs
Maximise Shannon’s Entropy Functional

\[ \max_P \left\{ H(P) = - \sum_{n=0}^{\infty} P(n) \log P(n) \right\} \]

subject to

- Normalisation, \( \sum_{n=0}^{\infty} P(n) = 1, \)
- Mean queue length, \( \sum_{n=0}^{\infty} np(n) = L \)
- Utilisation, \( \sum_{n=0}^{\infty} h(n)p(n) = 1 - p(0) = \rho, \quad \rho = \frac{\lambda}{\mu}, \quad 0 < \rho < 1 \)

\( h(n) = 1, \) if \( n = 0 \) or \( 0 \) otherwise

Apply the Method of Lagrange’s Undetermined Multipliers
A Stable G/G/1 Queue (Cont.)

- **A ME Generalised Geometric Solution**

\[
P(n) = \begin{cases} 
1 - \rho, & n = 0 \\
(1 - \rho)gx^n, & n \geq 1 
\end{cases}
\]

\[
g = \frac{\rho^2}{(L - \rho)(1 - \rho)}, \quad x = \frac{L - \rho}{L}
\]

where

\[
L = \frac{\rho}{2} \left(1 + \frac{1 + \rho C_s^2}{1 - \rho}\right)
\]

Mean queue length in M/G/1 queue (Pollaczek-Khintchine Formula)\(C_s^2\) : SCV of the service time, \(C_a^2 = 1\).
Maximise Shannon’s Entropy Functional

\[
\max_P \left\{ H(P) = -\sum_{n=0}^{\infty} P_N(n) \log P_N(n) \right\}
\]

subject to

- The normalization, \( \sum_{n=0}^{N} p_N(n) = 1 \)
- The mean queue length, \( \sum_{n=0}^{N} np_N(n) = L_N \)
- The utilisation, \( \sum_{n=0}^{N} h(n)p(n) = 1 - p_N(0) = U \)
- Full buffer state probability, \( \sum_{n=0}^{N} s(n)p_N(n) = \varphi = p_N(n), \ 0 < \varphi < 1 \)

where \( h(n) = 1, \text{ if } n = 0 \) or 0 otherwise and \( s(n)=1, \text{ if } n=N \) or 0 otherwise
**A Censored G/G/1/N Queue**

- Apply the Method of Lagrange’s Undetermined Multipliers
- Obtain a **Truncated** Generalised Geometric ME Solution (expressed in terms of the single step recursions) →

\[
P_N(1) = g x P_N(0)
\]

\[
P_N(n) = x P_N(n-1) \quad n = 2, \ldots, N - 1
\]

\[
P_N(N) = y x P_N(N-1)
\]

where

\[
g = \frac{\sigma \rho \tau}{\sigma \rho + \tau (1 - \sigma)}, \quad x = \frac{\sigma \rho + \tau (1 - \sigma)}{\sigma \rho + \tau (1 - \sigma)}, \quad y = \frac{\sigma \rho + \tau (1 - \sigma)}{\sigma + \tau (1 - \sigma)} \cdot \frac{1}{\tau},
\]

\[
\rho = \frac{\lambda}{\mu}, \quad \sigma = \frac{2}{C_a^2 + 1} \quad \text{and} \quad \tau = \frac{2}{C_s^2 + 1}
\]

The ME solution satisfies the flow balance condition

\[
\lambda (1-\pi) = \mu (1-P_N(0)), \quad \text{where} \; \pi \; \text{is the blocking probability.}
\]
Connection with the GE-type Distribution

**Theorem:** The ME M/G/1 solution is equivalent to the queue length distr. of a stable M/G/1 queue with a GE-type service time prob. density function of the form

\[
f(t) = (1 - r)u_0(t) + r^2 \mu e^{-r\mu t}, \quad t \geq 0,
\]

where \( r = \frac{2}{C_s^2 + 1} \), \( u_0(t) = + \infty, \) if \( t = 0 \) or, 0, if \( t \neq 0 \).

Unit impulse function

This theoretical result can be shown by substituting \( g, x \) and \( L \) into the ME solution and equating its z-transform with the Laplace-Stieltjes transform of the service time [Kouvatsos1994].
Connection with the GE-type Distribution

**Proof:** The Pollaczek-Khintchin z-transform of is

\[ Q(z) = \frac{F^*(\lambda - \lambda z)(1 - \rho)(1 - z)}{F^*(\lambda - \lambda z) - z} \]

where \( F^*(\theta) \) is the Laplace-Stieltjes transform of the service time. This transform can be determined directly using the relation

\[ Q(z) = \sum_{n=0}^{\infty} P(n)z^n, |z| \leq 1 \]

This implies that

\[ Q(z) = \frac{(1 - \rho)[1 - xz(1 - r)]}{(1 - xz)} \]
Connection with the GE-type Distribution

- It can be easily verified that $Q(0) = 1 - \rho$ and $Q(1) = 1$. Equating the right-hand sides of both equation, substituting for $x$ and $\rho$ and solving for $F^*(\lambda - \lambda z)$, the following result is obtained (with $r = 2 / (1 + C_s^2)$)

$$F^*(\lambda - \lambda z) = \frac{r \mu + (1-r)(\lambda - \lambda z)}{r \mu + \lambda - \lambda z}$$

- Substituting $\theta$ for $(\lambda - \lambda z)$, becomes

$$F^*(\theta) = \frac{r \mu + (1-r)\theta}{r \mu + \theta} = (1-r) + \frac{r^2 \mu}{r \mu + \theta}$$

- By inverting $F^*(\theta)$, the result follows. Q.E.D
The **GE-type Distribution**

- The ME solution of a stable M/G/1 queue is exact if $G \equiv GE$. Similarly for a stable GE/G/1 queue.
- The **GE-type** distr. with parameters $\alpha$ and $\beta$ ($0 \leq \alpha \leq 1$):

$$F(t) = 1 - \alpha e^{-\beta t}, \quad t \geq 0,$$

\[1 - \alpha = \frac{C^2 - 1}{C^2 + 1}\]
\[\alpha = \frac{2}{C^2 + 1}\]
\[\beta = \frac{2\nu}{C^2 + 1}\]

- The underlying counting process of the GE-type distr. is a compound Poisson process with Geo distributed batch sizes and mean batch size $1/\alpha = (C^2 + 1)/2$. 
**Interpretation of GE-type distribution**

- GE is an extremal case of the family of two-phase exponential distributions having the same \( \{v, C^2(>1)\} \)

- GE is a bulk type distribution with an underlying counting process equivalent to a **Compound Poisson Process (CPP)** with parameter \( 2v/(C^2 + 1) \) and a geometrically distributed bulk size with mean \( (1+C^2)/2 \) and SCV \( (C^2 - 1)/(C^2 + 1) \)  i.e.,

\[
P(N_{cp} = n) = \begin{cases} 
\sum_{i=1}^{n} \frac{\sigma^i}{i!} e^{-\sigma} \left( \frac{n-1}{i-1} \right) \tau^i (1-\tau)^{n-i}, & n \geq 1 \\
 e^{-\sigma}, & n = 0 
\end{cases}
\]

where \( N_{cp} \) is the random variable of the number of events per unit time.
Global Balance Solution for the Censored GE/GE/1/N Queue

- **GE-Type Algebra**
  - GE-type Transition Rates
  - GE-type Global Balance (GB) Equations
  - GE-type GB Solution for the State Probability Distribution
  - GE-Type GB Connection with ME Formalism
  - GE-type Blocking Probability
Global Balance (GB) Solution for the Censored GE₁/GE₂/1/N Queue

Let GE₁ ~ GE(σ,σλ) & GE₂ ~ GE(τ,τμ), where

- σ, τ are the stage selection probabilities of the non-zero exponential branches of the GE₁ and GE₂, respectively

- σλ, τμ are the arrival and service rates (at non-zero exponential branch of the queue), respectively i.e.,

\[
\sigma = \frac{2}{1+C^2a}
\]

\[
\tau = \frac{2}{1+C^2s}
\]
The analysis utilises the bulk interpretation of the GE-type distribution.

Suppose the number in the queue is $1 \leq k \leq N-1$ when a bulk of size $n \geq N-k$ arrives ➔ Implications

- Then $N-k$ units are chosen randomly from the bulk to fill the empty spaces of the waiting room.
- The remaining units of the bulk are considered to be lost.
GE-type GB Equations

\[
\sigma \lambda \frac{\tau}{\tau(1-\sigma) + \sigma} P_0 = \tau \mu \sum_{k=1}^{N} (1-\tau)^{k-1} P_k
\]

\[1 \leq i \leq N-1\]

\[(\sigma \lambda + \tau \mu) P_i = \sigma \lambda \frac{\tau \sigma(1-\sigma)^{i-1}}{\tau(1-\sigma) + \sigma} P_0 + \sigma \lambda \left( \sum_{k=1}^{i-1} \sigma(1-\sigma)^{i-k-1} \right) P_k + \tau \mu \left( \sum_{k=i+1}^{N} \tau(1-\tau)^{k-i-1} \right) P_k\]

\[
\tau \mu P_N = \sigma \lambda \frac{\tau(1-\sigma)^{N-1}}{\tau(1-\sigma) + \sigma} P_0 + \sigma \lambda \left( \sum_{k=1}^{N-1} (1-\sigma)^{N-k-1} \right) P_k
\]
The GE-type GB State Probability Distribution

\[ P_N = P_0 \frac{\sigma \rho}{\tau (1 - \sigma) + \sigma} \left( \frac{\sigma \rho + \tau (1 - \sigma)}{\sigma \rho + \tau (1 - \sigma \rho)} \right)^{N-1}, \quad \rho = \frac{\lambda}{\mu} \]

\[ P_k = P_0 \frac{\sigma \rho \tau}{\sigma \rho + \tau (1 - \sigma \rho)} \left( \frac{\sigma \rho + \tau (1 - \sigma)}{\sigma \rho + \tau (1 - \sigma \rho)} \right)^{k-1}, \quad 1 \leq k \leq N - 1 \]

\[ P_0 = \frac{1 - \rho}{1 - \rho \frac{\sigma \rho + \tau (1 - \sigma \rho)}{\sigma + \tau (1 - \sigma)} \left( \frac{\sigma \rho + \tau (1 - \sigma)}{\sigma \rho + \tau (1 - \sigma \rho)} \right)^N} \]
**GB Connection with the ME Formalism**

- **Maximise Entropy Functional**

\[
\max_P \left\{ H(P) = -\sum_{n=0}^{\infty} P_N(k) \log P_N(k) \right\}
\]

subject to normalisation, utilisation, mean queue length and full buffer state probability constraints satisfying the flow balance Condition: \( \lambda(1-\pi) = \mu(1-PN(0)) \), where \( \pi \) is the blocking probability.

- **ME Solution**

\[
P_N(1) = g x P_N(0)
\]

\[
P_N(k) = x P_N(k-1) \quad k = 2, \ldots, N-1
\]

\[
P_N(N) = y x P_N(N-1)
\]

where

\[
g = \frac{\sigma \rho \tau}{\sigma \rho + \tau (1-\sigma)} , \quad x = \frac{\sigma \rho + \tau (1-\sigma)}{\sigma \rho + \tau (1-\sigma)} , \quad y = \frac{\sigma \rho + \tau (1-\sigma)}{\sigma + \tau (1-\sigma)} \frac{1}{\tau}
\]

\[
\rho = \frac{\lambda}{\mu} , \quad \sigma = \frac{2}{C_{a}^2 + 1} \quad \text{and} \quad \tau = \frac{2}{C_{s}^2 + 1}
\]
The GE-type Blocking Probability

The probability of an arrival to find the queue full, \( \pi \), is given by

\[
\pi = P_N(N) + \sum_{k=1}^{N-1} P_N(k)(1-\sigma)^{N-k} + P_N(0) \frac{\tau(1-\sigma)^N}{\tau(1-\sigma) + \sigma}
\]

\[
= \sum_{k=0}^{N} \delta(k) P_N(k)(1-\sigma)^{N-n}
\]

where

\[
\delta(k) = \begin{cases} 
\frac{\tau}{\tau(1-\sigma) + \sigma} & \text{if } k=0 \\
1 & \text{if } k \neq 0
\end{cases}
\]

The proof is based on the bulk interpretation of the compound Poisson arrival process to the queue and the GE-type service time distribution.
The GE-type Blocking Probability (cont.)

- The bulk finds N jobs in the GE/GE/1/N queue; Bulks arrive according to a Poisson ($\sigma \lambda$) process. Thus a tagged arriver will find N in the system with probability $P_N(N)$ (i.e., the same with that of a random observer).

- The bulk finds k jobs in GE/GE/1/N the queue ($1 \leq k \leq N-1$); The size of the bulk is at least $m = N-k+1$ and the tagged arriver is one of those bulk members that will be blocked (turned away). Thus,

$$
\sum_{k=1}^{N-1} P_N(k) \left( \sum_{m=N-k+1}^{\infty} m \frac{\sigma(1-\sigma)^{m-1}}{1/\sigma} \right) \frac{m-(N-k)}{m} = \sum_{k=1}^{N-1} P_N(k) (1-\sigma)^{N-k}
$$
The bulk finds 0 jobs the GE/GE/1/N queue;

The bulk size $m$ is at least $(N+1)$, at most $m-(N+1)$ jobs choose the null GE branch from the front part of the bulk and the tagged arriver is one of those bulk units that will be blocked (turned away). Thus,

$$P_N(0) \sum_{m=N+1}^{\infty} \frac{m\sigma(1-\sigma)^{m-1}}{1/\sigma} \sum_{k=0}^{m-(N+1)} \tau(1-\tau)^k \frac{m-N-k}{m} = P_N(0) \frac{\tau(1-\sigma)^N}{\tau(1-\sigma)+\sigma}$$

The form of the GE-type blocking probability, $\pi$, of the GE/GE/1/N queue is obtained by adding the probabilities of these three mutually exclusive events.
The Havrda-Charvat generalised parametric entropy function, $Sq$, is defined by [Havrda & Charvat 1967]

$$S_q = \frac{C(1 - \sum_{i=0}^{\infty} p_i^q)}{q - 1}$$

$p_i$, $i=0,1,…$ are the state probabilities of the queue;

$q$ is a real number measuring the degree of non-extensivity of the queue;

$C$ is a positive constant;

$Sq$ is a generalised measure of uncertainty in dynamic systems, which reduces to Shannon entropy function at the non-extensivity parameter $q \to 1$ limit $H$. 

$Havrda-Charvat Generalised Entropy Function$
Generalised Entropy Maximisation: Generalization of Boltzmann Gibbs Statistics

In Statistical Mechanics, Tsallis (1988) proposed independently an equivalent to Havrda-Charvat entropy function

$$S_q = \frac{C \left( 1 - \sum_{i=0}^{\infty} p_i^q \right)}{q - 1}$$

which was maximised subject to:

1. $\sum_{i=1}^{W} p_i = 1$
2. $\sum_{i=1}^{W} \varepsilon_i p_i = U_q$

where $W$ is the no. of microscopic configurations and $\{\varepsilon_i, U_q\}$ are known as generalized spectrum and generalized internal energy.

Maximisation of $S_q$ gives a Zipf-Mandelbrot power-type distribution with non-extensive properties.
Tsallis (1988) Solution

- Introduce $\alpha$ and $\beta$ Lagrange multipliers and define the quantity

$$\phi_q = \frac{S_q}{C} - \alpha \sum_{i=1}^{W} p_i - \alpha \beta (q - 1) \sum_{i=1}^{W} \varepsilon_i p_i$$

by taking $\frac{\partial \phi_q}{\partial p_i} = 0$, one obtains $p_i = \frac{[1 - \beta (q - 1) \varepsilon_i]}{Z_q}^{1/q-1}$

where $Z_q = \sum_{i=1}^{W} [1 - \beta (q - 1) \varepsilon_i]^{1/q-1}$

At the $q \to 1$ limit,

$$p_i = e^{-\beta \varepsilon_i} \sqrt{Z}$$, with $Z = \sum_{i=1}^{W} e^{-\beta \varepsilon_i}$

i.e., solution of M/M/1 queue [Assi 2000, Kouvatsos and Assi 2002, Karmeshu & Sharma 2005]
G/G/1 Queue: An EME Framework

- Maximise Generalised Entropy Functional

\[
\max_P \left\{ S_q = \frac{C \left(1 - \sum_{i=0}^{\infty} \rho(n)^q\right)}{q-1} \right\}
\]

subject to

- The normalization, \( \sum_{n=0}^{\infty} \rho(n) = 1 \)
- The mean queue length, \( \sum_{n=0}^{\infty} np(n) = L \)
- The utilisation, \( \sum_{n=0}^{\infty} h(n)\rho(n) = 1 - \rho(0) = \rho, \quad \rho = \frac{\lambda}{\mu}, \quad 0 < \rho < 1 \)

where \( h(n) = 1, \) if \( n = 0 \) or otherwise.

Apply the Method of Lagrange’s Undetermined Multipliers

[Assi 2000], [Kouvatsos & Assi 2002, 2007]
G/G/1 Queue: An EME Framework

A Generalised Zipf-Mandelbrot EME power-type distribution

\[ p(n) = \frac{1}{\sum_{n=0}^{\infty} \left[ 1 + \alpha(1 - q)n + \beta(1 - q)h(n) \right]^{\frac{1}{q-1}}} \left[ 1 + \alpha(1 - q)n + \beta(1 - q)h(n) \right]^{\frac{1}{q-1}}, \quad n = 0, 1, \ldots \]

- At the \( q \rightarrow 1 \) limit,

\[ p(n) = \frac{e^{-\lambda n - \beta h(n)}}{Z} = \frac{x^n g^{h(n)}}{Z}, \quad \text{with} \quad Z = \sum_{n=0}^{\infty} x^n g^{h(n)}, \quad x = e^{-\lambda}, \quad g = e^{-\beta} \]

ME state probability distribution of a stable G/G/1 queue

- \( x \) and \( g \) are the Lagrangian coefficients corresponding to mql and server utilisation constraints. Moreover, \( \frac{1}{2} < q < 1 \).
**G/G/1/N Queue: An EME Framework**

- **Maximise Generalised Entropy Functional**

\[
\max_P \left\{ S_q = \frac{C \left( 1 - \sum_{i=0}^{\infty} p_N(n)^q \right)}{q - 1} \right\},
\]

subject to:

- The normalization, \( \sum_{n=0}^{N} p_N(n) = 1 \)
- The mean queue length, \( \sum_{n=0}^{N} np_N(n) = L_N \)
- The utilisation, \( \sum_{n=0}^{N} h(n)p(n) = 1 - p_N(0) = U \)
- Full buffer state probability, \( \sum_{n=0}^{N} s(n)p_N(n) = \varphi = p_N(n), \quad 0 < \varphi < 1 \)

where \( h(n) = 1, \) if \( n = 0 \) or 0 ow. and \( s(n) = 1, \) if \( n = N \) or 0 ow., satisfying the flow balance condition: \( \lambda \left( 1 - \pi \right) = \mu(1 - P_N(0)) \)

A **Truncated Generalised Zipf-Mandelbrot** EME power-type distribution

\[
p_N(n) = \frac{\left[1 + \alpha(1 - q)n + \beta(1 - q)h(n) + \gamma(1 - q)s(n)\right]^1}{\sum_{n=0}^{N}\left[1 + \alpha(1 - q)n + \beta(1 - q)h(n) + \gamma(1 - q)s(n)\right]^1}
\]

At the \( q \to 1 \) limit,

\[
p_N(n) = \frac{e^{-\alpha n - \beta h(n) - \gamma s(n)}}{\sum_{n=0}^{N}e^{-\alpha n - \beta h(n) - \gamma s(n)}} = \frac{x^ng^{h(n)}y^{s(n)}}{\sum_{n=0}^{N}x^ng^{h(n)}y^{s(n)}}, \quad n = 0, 1, \ldots, N
\]

\[x = e^{-\alpha}, \quad g = e^{-\beta}, \quad y = e^{-\gamma}\]

This is the corresponding known solution of a **GE/GE/1/K** queue.

For \( q < 1 \) and for large number of jobs \( n \), the EME solution follows the **power law**:

\[p_N(n) \sim n^{1/q-1}, \quad 1/2 < q < 1\]
A heuristic relation between the non-extensivity parameter, $q$ and the Hurst parameter, $H$ can be achieved by using the boundary conditions

The boundary conditions of the non-extensivity parameter $q$ of Tsallis entropy solution is $\frac{1}{2} < q < 1$;

The boundary conditions of Hurst parameter $H$ of the fractional Brownian Motion (fBm) is $\frac{1}{2} < H < 1$;

It is implied that for $q \to 1$ (Shannon’s Entropy) $\Rightarrow$ $H \to 0.5$ (exponential distribution)

for $q \to 1/2$ (max value of non extensivity parameter) $\Rightarrow$ $H \to 1$ (Pareto distribution with power law tails corresponding to max value of $H$)

The following simple heuristic relationship is usually defined

$$H = 1.5 - q$$

[Karmeshu & Sharma 2005], [Kouvatsos & Assi 2007]
In the context of the EME approach, a mean queue length constraint for a fBM/M/1 queue (c.f., [Karmeshu and Sharma 2005], [Kouvatsos & Assi 2007]) was motivated by a reinterpretation of a formula proposed in Norros [1994] in the context of ATM networks, for calculating buffer capacity of a simple storage model with self-similar input traffic process modelled by a fBm as an input process with Hurst parameter, $H$, $H \in [0.5,1]$ and exponential service time:

$$< n > = \frac{\rho}{(1 - \rho)^{H/(1-H)}}^{1/[2(1-H)]}, \quad 0.5 < H < 1, \quad \rho = \frac{\lambda}{\mu}$$

where $\lambda$ and $\mu$ are the mean arrival and service rates, respectively.
A Heuristic Generalisation for an EME Mean Queue Length

A heuristic extension of Norros formula [Norros 1994] was conjectured in [Kouvatsos & Assi 2007] for calculating the buffer capacity of a simple storage model with generalised fractional Brownian motion (gfBm) process as an input traffic and GE-type service time distribution, namely

\[ <n> = \frac{\rho^{\frac{1}{2(1-H)}}}{2^{\frac{1}{2(1-H)}}} \left( \frac{1 - \rho + C_a^2 + \rho C_s^2}{(1 - \rho)^{H/(1-H)}} \right)^{\frac{1}{2(1-H)}}, \quad 0.5 < H < 1, \quad \rho = \frac{\lambda}{\mu} \]

where \( C^2 a \) and \( C^2 s \) are the interarrival time and service time SCV and \( H \) is the Hurst parameter taking values in the interval \([1/2, 1]\).

For the computational implementation of the EME solutions, the generalised formula is adopted as the mql, \( E(N) \), of a stable infinite capacity gfBm/GE/1 queue.

For \( H = \frac{1}{2} \), it yields the result for mean queue length of a stable \( GE/GE/1 \) queue which corresponds to the case \( q \to 1 \) in the proposed framework.
Overflow Probability

- The probability distribution for the queue length distribution \{p_N(n), n=0,1,...\} can be rewritten in terms of Hurwitz-Zeta function as,

\[
p_K(n) = \left[ \frac{1 + \beta(1-q)h(n) + \gamma(1-q)s(n)}{\alpha(1-q)} + n \right] \cdot \frac{1}{q^{-1}} \cdot \zeta \left[ \frac{1}{1-q}, \frac{1 + \beta(1-q)h(n) + \gamma(1-q)s(n)}{\alpha(1-q)} \right], \{q > 0, n = 0,1,....K\}
\]

- The overflow probability,

\[
P(n > x) = \left\{ \int_{0}^{\left( 1 + \frac{1 + \beta(1-q)h(n) + \gamma(1-q)s(n)}{\alpha(1-q)} \right)} \frac{1}{q^{-1}} \cdot \zeta \left[ \frac{1}{1-q}, \frac{1 + \beta(1-q)h(n) + \gamma(1-q)s(n)}{\alpha(1-q)} \right] \cdot \left( x + \frac{1 + \beta(1-q)h(n) + \gamma(1-q)s(n)}{\alpha(1-q)} \right)^{-\frac{q}{1-q}} \right\}
\]

\[
\cdot \left\{ \int_{0}^{\left( 1 + \frac{1 + \beta(1-q)h(n) + \gamma(1-q)s(n)}{\alpha(1-q)} \right)} \frac{1}{q^{-1}} \cdot \zeta \left[ \frac{1}{1-q}, \frac{1 + \beta(1-q)h(n) + \gamma(1-q)s(n)}{\alpha(1-q)} \right] \cdot \left( x + \frac{1 + \beta(1-q)h(n) + \gamma(1-q)s(n)}{\alpha(1-q)} \right)^{-\frac{q}{1-q}} \right\}
\]

\[
\cdot \frac{1}{\zeta \left[ \frac{1}{1-q}, \frac{1 + \beta(1-q)h(n) + \gamma(1-q)s(n)}{\alpha(1-q)} \right]} \cdot \frac{1}{\left( 1 + \frac{1 + \beta(1-q)h(n) + \gamma(1-q)s(n)}{\alpha(1-q)} \right)} ^{\frac{q}{1-q}} \cdot \left( \sum_{n=0}^{N} \int_{u=n}^{1} u \cdot \left( u + n + \frac{1 + \beta(1-q)h(n) + \gamma(1-q)s(n)}{\alpha(1-q)} \right)^{-\frac{q}{1-q}} \right) du
\]
Overflow Probability

- For asymptotically large $x$ a power law is determined by,

$$P(n > x) \sim Wx^{-q/(1-q)}, \quad W = \left(1/s\right)\left[\frac{1}{1-q}, \frac{1+\beta(1-q)h(n)+\gamma(1-q)s(n)}{\alpha(1-q)}\right]\left(\frac{1-q}{q}\right)$$

- at the $q \to 1$ limit,

$$P(n > x) \sim e^{-\alpha x - \beta h(n) - \gamma s(n)}$$
Server Utilisation and Blocking Probability

- The probability that server is busy (i.e., the server utilization, \( U \))

\[
U = 1 - p_K(0)
\]

\[
= 1 - \left[ \frac{\alpha(1-q) + \beta(1-q)h(n) + \gamma(1-q)s(n)}{\zeta \left( \frac{1}{1-q}, \frac{1 + \beta(1-q)h(n) + \gamma(1-q)s(n)}{\alpha(1-q)} \right)} \right]^{\frac{1}{q-1}}
\]

- Using the flow balance condition, the blocking probability can be obtained by

\[
\pi = 1 - \frac{U}{\rho}
\]

Note: All these formulae together with the associated algorithms below can be found in [Kouvatsos & Assi 2007] & are generalisations to those reported in [Karmeshu & Sharma 2005].
EME Analytic Algorithms

EME ALGORITHM 1: The gfBm /GE/1 Queue

- **Input Data** \{ q, \lambda, C_a^2, \mu, C_s^3 \}

- Begin

- **Step 1** Calculate \( H = 1.5 - q \) and mean queue length, \( <n> \)

- **Step 2** Set initial approximations of Lagrangian multipliers \{ \alpha, \beta \}.

- **Step 3** Solve constraints (2) and (3) via Newton-Raphson method wrt \{ \alpha, \beta \}.

- **Step 4** Obtain new values for \{ \alpha, \beta \}.

- **Step 5** Return to Step 3 until convergence of \{ \alpha, \beta \}.

- End

- **Output Statistics:** The Lagrange’s multipliers \{ \alpha, \beta \} and state probabilities, \{p(n)\}.
EME ALGORITHM 2: The Censored gfBm /GE/1/N Queue

- **Input Data** \{ N, \alpha, \beta, q, \lambda, C_a^2, \mu, C_s^2 \}
- **Begin**
- **Step 1** Initial approximation of Lagrangian multiplier \( \gamma \);
- **Step 2** Solve constraints (1) and (4) using the Newton-Raphson method wrt \( \eta \);
- **Step 3** Obtain new values for \( \gamma \);
- **Step 4** Return to Step 2 until convergence of \( \gamma \);
- **Step 5** Using flow balance condition to compute blocking probability.
- **End**
- **Output Statistics**: The Lagrangian multipliers, \( \gamma \), state probability (\( P_N(n) \)) and the blocking probability, \( \pi \)
The relation between $p_N(n)$ and $n$ for a finite capacity queue with $(q = 0.5, 0.6, 0.7, 0.9)$, $C^2a = 8$, $C^2s = 4$, $\lambda=0.03$, $\mu=0.2$ and $N =30$. 

**Numerical Results**

![Graph showing the relation between $p_N(n)$ and $n$ for various values of $q$.](image)
The relation between the queue length distribution and $n$ for a finite capacity queue with $\lambda=0.02$, $\mu=0.4$, $q = 0.6$, $C^2s = 4$ and $N =30$
The relation between $\rho$ and $U$ (Utilisation) with $\{C^2 a = 1, 5 \& 10\}$, $C^2 s = 3$, $q = 0.6$ and $N = 20$. 

Numerical Results
Numerical Results

The relation between $\rho$ and $U$ (Utilisation) with $C^2a = 3$, $C^2s = 4$, $q = \{0.6, 0.7, 0.8, 1\}$
Numerical Results

The relation between traffic intensity, $\rho$ and server utilisation, $U = 1 - P(0)$ for a finite capacity GfBm/GE/1/K queue with $Ca^2 = 20$, $Cs^2 = 3$ and $q = 0.7, 0.8, 0.9$. 
The relation between the mean queue length and $N$ (queue capacity) with $C^2 a = 4$, $C^2 s = 9$, $\lambda = 0.1$, $\mu = 0.5$ and $(q = 0.6, 0.7, 0.8, 0.9)$
Conclusions & Extensions to
Arbitrary Open Queueing Network Models (QNMs)

- Product-Form Approximations and
  Queue-by-Queue Decomposition
  of Arbitrary Open Queueing Network Models
  (QNMs) with Blocking

[Kouvatsos & Awan 2003]
Queue-by-Queue Decomposition of Open QNMs
Some Typical References


Some Typical References


Some Typical References


Some Typical References


Some Typical References